On Galilei-covariant Lagrangian models of fluids

2001 J. Phys. A: Math. Gen. 34 10921

(http://iopscience.iop.org/0305-4470/34/49/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10921–10937 PII: S0305-4470(01)28121-9

On Galilei-covariant Lagrangian models of fluids

M de Montigny[1,2], **F C Khanna**[2,3] **and A E Santana**[2,4]

[1] Faculté Saint-Jean, University of Alberta, 8406-91 Street, Edmonton, Alberta, Canada T6C 4G9
[2] Theoretical Physics Institute, University of Alberta Edmonton, Alberta, Canada T6G 2J1
[3] TRIUMF, 4004, Wesbrook Mall, Vancouver, British Columbia, Canada V6T 2A3
[4] Instituto de Física, Universidade Federal da Bahia, Campus de Ondina, Salvador, Bahia, 40210-340, Brazil

E-mail: montigny@phys.ualberta.ca, khanna@phys.ualberta.ca and santana@fis.ufba.br

Received 14 August 2001
Published 30 November 2001
Online at stacks.iop.org/JPhysA/34/10921

Abstract

We use a covariant-like formulation of Galilei invariance in five dimensions to obtain a model for compressible irrotational barotropic fluids with pressure proportional to the square of the mass density. Some solutions for the one-dimensional version of one of these equations are found by using their Lie point symmetries. Other models of fluids are also discussed. Our purpose is to illustrate how the Galilei-covariant formalism can serve as a guide for the construction of non-relativistic wave equations relevant in many-body theories.

PACS numbers: 11.10.Ef, 05.60.Gg, 11.30.-j, 45.20.Jj, 67.40.Pm

1. Introduction

The problem of deriving transport (or fluid) equations has developed over the years with the evolution of ideas and concepts in physics. A trend in treating such problems has been adopting the notion of a field as a fundamental concept to describe the state of a fluid. The analysis of a fluid system has then been improved in at least two distinct directions. On the one hand, there are methods based on stochasticity [1, 2]. In this case, the usual notion of a fluid considered as a set of individual molecules, with movement strictly controlled by classical or quantum mechanics, has been enlarged to encompass stochastic ingredients in the fluid state. On the other hand, the problem has been addressed via the theory of symmetry groups and geometry [3–6]. In this case, the state of a fluid is described by elements of the group-representation carrier space.

Following the concept of symmetry, Takahashi, for instance, has undertaken the investigation of the non-relativistic (classical and quantum) field theory using the Galilei invariance as a guide for constructing the underlying models [5, 6]. A remarkable consequence of this work was the foundation of a manifestly covariant Galilean approach, i.e. based on tensor calculus. This method, introduced via embedding of the usual Newtonian space-time into a five-dimensional de Sitter space, was investigated more systematically later [7, 8]. Although the idea of using a five-dimensional space already existed [9] in the literature, it was not exploited in full (for example, see [10, 11]).

Motivation for the Galilei-covariant approach was provided by the study of liquid ^{4}He, in particular its superfluidity at low temperatures. In the Landau theory for superfluids [12], elementary excitations, the so-called rotons and phonons, emerge as a consequence of the rearrangement of the original (Galilei-invariant) Schrödinger field [13]. Moreover, the existence of a superfluid state occurs only in the realm of low velocities. All of these features make the Galilei transformations a central element in the Landau formalism. This observation led Takahashi to derive, by using Galilei invariance, a non-linear equation which is claimed to be associated with these elementary excitations. Our goal here is to explore, in particular, such an equation, but with a new ingredient: a manifestly Galilean-covariant approach for the non-relativistic physics. This formalism is reviewed in section 2. We introduce in section 3 a scalar Galilei-covariant Lagrangian with an interaction term, and thereby we proceed by using the standard techniques of field theory. In section 4, the balance equations for the system are derived from the divergenceless energy–momentum tensor, and the Euler–Lagrange equation provides a parameter-dependent class of equations which has the Takahashi equation as a particular case. In section 5, we use the methods of Lie group theory applied to partial differential equations to find particular group-invariant solutions.

2. Outline of Galilei covariance

Let us review briefly the Galilean tensor formalism [7] and set up the notation. It is based on a five-dimensional space such that a Galilean boost acts on a vector $(\boldsymbol{x}, t, s)$ as

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{x} - \boldsymbol{V}t \\ t' &= t \\ s' &= s - \boldsymbol{V}\cdot\boldsymbol{x} + \tfrac{1}{2}v^2 t \end{aligned} \tag{1}$$

with relative velocity $\boldsymbol{V}$. The dimensions of the components are

$$\begin{aligned} [\boldsymbol{x}] &= L \text{ (length)} \\ [t] &= T \text{ (time)} \\ [s] &= \frac{L^2}{T} \end{aligned} \tag{2}$$

so a five-vector $(x^1, \ldots, x^5)$, where each component has the dimension of length, can be defined from equation (1) as

$$(x^1, \ldots, x^5) = \left(\boldsymbol{x}, v_4 t, \frac{s}{v_5}\right) \tag{3}$$

where $[v_4] = [v_5] = L/T$. This vector transforms as

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{x} - \frac{\boldsymbol{V}}{v_4}x^4 \\ x'^4 &= x^4 \\ x'^5 &= x^5 - \frac{1}{v_5}\boldsymbol{V}\cdot\boldsymbol{x} + \frac{1}{2}\frac{v^2 x^4}{v_4 v_5}. \end{aligned} \tag{4}$$

The scalar product

$$(A|B) = A^\mu B_\mu \equiv \boldsymbol{A}\cdot\boldsymbol{B} - A_4 B_5 - A_5 B_4 \tag{5}$$

of two five-vectors A and B is clearly invariant under equation (1). This suggests basing the Galilean tensor calculus on the metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6}$$

Hereafter we refer to this metric as the *Galilean metric*.

Perhaps the easiest way to explain the motivation for the transformation of the parameter s in equation (1) is to analyse a Galilean boost in the momentum space. Consider a single free particle of mass m with velocity $\boldsymbol{v}$ in some reference frame. Then its momentum is $\boldsymbol{p} = m\boldsymbol{v}$ and its energy is $E = \frac{1}{2m}\boldsymbol{p}^2$. In a frame moving with a relative velocity $\boldsymbol{V}$, the momentum is

$$\boldsymbol{p}' = m(\boldsymbol{v} - \boldsymbol{V}) = \boldsymbol{p} - m\boldsymbol{V} \tag{7}$$

so the energy is given by

$$\begin{aligned} E \to E' &= \frac{1}{2m}(\boldsymbol{p}')^2 \\ &= \frac{1}{2m}(\boldsymbol{p} - m\boldsymbol{V})^2 \\ &= \frac{\boldsymbol{p}^2}{2m} - \boldsymbol{p}\cdot\boldsymbol{V} + \frac{1}{2}m\boldsymbol{V}^2 \\ &= E - \boldsymbol{p}\cdot\boldsymbol{V} + \tfrac{1}{2}m\boldsymbol{V}^2. \end{aligned} \tag{8}$$

Noting that the mass is invariant, that is,

$$m' = m \tag{9}$$

one may conclude that $(\boldsymbol{p}, m, E)$ is a five-vector which transforms as equation (1) with $\boldsymbol{p}$, m and E replaced by $\boldsymbol{x}$, t and s, respectively. (Another argument in favour of the introduction of an additional parameter, using the quasi-invariance of a Lagrangian under a Galilean boost, can be found in [7, 14].) The Galilei-covariant formulation consists in building Lagrangian models in a five-dimensional space endowed with the Galilean metric. The resulting equations are then projected onto the Newtonian space through an *embedding*, i.e. some restrictions on the coordinates (or momenta) and the fields. A general five-momentum $(p^1, \ldots, p^5)$ transforms as $(\boldsymbol{p}, m, E)$ in equations (7)–(9) and lives in a genuine five-dimensional space. An embedding is performed, for instance, by selecting a four-dimensional surface such that $p^5 = \boldsymbol{p}^2/2p^4$. This is actually what was done above in order to obtain equation (8); but *a priori* the variable E can be independent of the others. In this sense the fifth coordinate s can be interpreted physically as the canonical conjugate to $p_5 \propto m$, in the same way as $\boldsymbol{x}$ is conjugate to $\boldsymbol{p}$ and t to E. (Note that the Galilean metric interchanges the covariant and the contravariant components: $p^4 \to -p_5$, $p^5 \to -p_4$.)

The choice of the embedding is crucial for determining the ensuing non-relativistic equations. Examples of embeddings that appear in the literature are

$$\begin{aligned} &\mathcal{E}_1: \qquad \boldsymbol{A} \to A = \left(\boldsymbol{A}, A_4, \frac{\boldsymbol{A}^2}{2A_4}\right) \\ &\mathcal{E}_2: \qquad \boldsymbol{A} \to A = (\boldsymbol{A}, A_4, 0) \\ &\mathcal{E}_3: \qquad \boldsymbol{A} \to A = \left(\boldsymbol{A}, \frac{A_4}{\sqrt{2}}, \frac{A_4}{\sqrt{2}}\right) \end{aligned} \tag{10}$$

where $\boldsymbol{A}$ belongs to the Newtonian space and A is in the five-dimensional space endowed with the Galilean metric [8]. The embedding $\mathcal{E}_1$ has also been used in [3].

In addition to $x^\mu = (\boldsymbol{x}, t, s)$ other Galilei vectors are $p^\mu = (\boldsymbol{p}, m, E)$ where $E = \frac{1}{2m}\boldsymbol{p}^2$, as well as $\partial^\mu = (\boldsymbol{\nabla}, -\partial_s, -\partial_t)$, $k^\mu = (\boldsymbol{k}, 0, \omega)$ etc. From these and equation (5), we can determine Galilei invariants such as $x^\mu x_\mu = \boldsymbol{x} \cdot \boldsymbol{x} - 2st$, $p^\mu p_\mu = \boldsymbol{p}^2 - 2mE$, $\partial^\mu \partial_\mu = \boldsymbol{\nabla}^2 - 2\partial_t \partial_s$, $k^\mu p_\mu = \boldsymbol{k} \cdot \boldsymbol{p} - m\omega$ etc. It is to be noted that in Lorentz-invariant theory, $E^2 - p^2$ is an invariant, equal to m^2, whereas in Galilean-invariant theory there is no such relation. However, one can take the invariant $\boldsymbol{p}^2 - 2mE$ to be equal to a Galilei scalar k^2 which plays a role analogous to the rest mass in Einstein's relativity. For a free particle, one has $k^2 = 0$.

Finally let us mention that, from a group theoretical point of view, the central extension of the Galilei group emerges as a subgroup of the Poincaré group in 4+1 dimensions which leaves the scalar product, equation (5), invariant. This Poincaré algebra admits fifteen generators, among which eleven elements generate the extended Galilei group with the central charge being the remnant of the generator of translations in the fifth dimension [8].

3. A Galilei-covariant Lagrangian for fluids

Hereafter we illustrate the formalism described above by recovering a model for the hydrodynamics of a compressible irrotational fluid with the barotropic condition $p \propto \rho^2$ where p is the pressure and ρ is the density of mass [5]. Hereafter we consider a classical field theory defined by a manifestly Galilei-covariant Lagrangian for a scalar field $\tilde{\phi}$ with a potential proportional to the square of the kinetic energy term, i.e. $(\partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi})^2$. Let us write this Lagrangian as

$$\mathcal{L} = \frac{\rho_0}{8v_0^2}\left(\partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi} - 2v_0^2\right)^2 \tag{11}$$

where ρ_0 and v_0^2 are constants that guarantee the coherence of units. The Lagrangian must satisfy the Euler–Lagrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \tilde{\phi})} - \frac{\partial \mathcal{L}}{\partial \tilde{\phi}} = 0 \tag{12}$$

where μ runs from 1 to 5, and $\tilde{\phi} = \tilde{\phi}(x)$ with x a five-vector as in equation (3). The various dimensions are

$$\begin{aligned}
&[\mathcal{L}] = \frac{M}{LT^2} \qquad (M : \text{mass}) \\
&[\partial] = \frac{1}{L} \\
&[\rho_0] = \frac{M}{L^3} \\
&[\tilde{\phi}] = \frac{L^2}{T} \\
&[v_0^2] = [\partial\tilde{\phi}\partial\tilde{\phi}] = \frac{L^2}{T^2}.
\end{aligned} \tag{13}$$

The field $\tilde{\phi}$ is a Galilei scalar; we will see in equation (29) that it is a potential for the velocity of the fluid.

The equation of motion for the field $\tilde{\phi}$ is

$$\partial_\mu \partial^\mu \tilde{\phi} - \frac{1}{2v_0^2}(\partial_\mu \partial^\mu \tilde{\phi})(\partial_\nu \tilde{\phi} \partial^\nu \tilde{\phi}) - \frac{1}{v_0^2}(\partial^\mu \tilde{\phi})(\partial^\nu \tilde{\phi})(\partial_\mu \partial_\nu \tilde{\phi}) = 0. \tag{14}$$

The corresponding five-dimensional energy–momentum tensor is

$$
\begin{aligned}
T^{\mu\nu} &= g^{\mu\nu}\mathcal{L} - \partial^\mu\tilde{\phi}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\tilde{\phi})} \qquad \mu,\nu = 1,\ldots,5 \\
&= \frac{\rho_0 g^{\mu\nu}}{8v_0^2}\left(\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - 2v_0^2\right)^2 - \frac{\rho_0}{2v_0^2}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi}\left(\partial^\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - 2v_0^2\right)
\end{aligned}
\tag{15}
$$

where $g^{\mu\nu}$ is the Galilean metric. It is easy to check that

$$
\partial_\mu T^{\mu\nu} = 0 \tag{16}
$$

as usual. It is clearly a symmetric tensor. The physical interpretation of each component is given at the end of the next section.

Other Lagrangians similar to equation (11) also lead to well-known equations of physics when analysed with the appropriate embeddings. A generalization of equation (11) is

$$
\mathcal{L} \propto \left(\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} + \text{constant}\right)^p \tag{17}
$$

where p is a rational number. There are two important classes of embeddings. One is embeddings such as $x^4 \propto t$, $x^5 \propto s$ and $\tilde{\phi} \approx \phi + s$, which leads to

$$
\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} \rightarrow \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \text{constant}\times\partial_t\phi. \tag{18}
$$

This embedding is used in this paper. Another class of embeddings with $x^4 \propto x^5 \propto t$ and $\tilde{\phi} \approx \phi + s$ is such that

$$
\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} \rightarrow \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \text{constant}\times(\partial_t\phi)^2. \tag{19}
$$

For instance, equation (17) with $p = 1$ and an embedding leading to equation (19) generates the Klein–Gordon equation and the like, e.g. the sine–Gordon equation. The sine–Gordon equation has been used for instance in elementary particle theory [15], Bloch wall motion of magnetic crystals [16] and propagation of magnetic flux on a Josephson line [17]. A similar embedding with $p = 1/2$ leads to the Born–Infeld equation. The three-dimensional version of the Born–Infeld equation was first investigated as a non-linear modification of the Maxwell equations to permit the electron to appear in a natural way as a singularity [18]. Note that in all cases the embedding at the level of the field is also crucial for determining the equations of motion.

4. The Takahashi model for fluids

Our purpose in this section is to recover a model introduced by Takahashi in [5]. We use the equation of motion, equation (14), with the embedding of the coordinates and field defined as

$$
x^4 \equiv v_4 t \qquad x^5 \equiv \frac{s}{v_5} \qquad \tilde{\phi}(x) \equiv \phi(\boldsymbol{x},t) + a_0 s \tag{20}
$$

where both v_4 and v_5 have units of velocity and a_0 is a dimensionless constant since $\tilde{\phi}$, ϕ and s have dimensions $\frac{L^2}{T}$. From this embedding we have

$$
\begin{aligned}
&\boldsymbol{\nabla}\tilde{\phi} = \boldsymbol{\nabla}\phi \qquad \partial_4\tilde{\phi} = \frac{1}{v_4}\partial_t\phi \qquad \partial_5\tilde{\phi} = a_0 v_5 \qquad \nabla^2\tilde{\phi} = \nabla^2\phi \\
&\partial_4\partial_5\tilde{\phi} = \partial_5\partial_4\tilde{\phi} = 0 \qquad \partial_5\partial_5\tilde{\phi} = 0 \qquad \partial_4\partial_4\tilde{\phi} = \frac{1}{v_4^2}\partial_t^2\phi \\
&\partial_4(\boldsymbol{\nabla}\tilde{\phi}) = \frac{1}{v_4}\partial_t(\boldsymbol{\nabla}\phi) \qquad \partial_5(\boldsymbol{\nabla}\tilde{\phi}) = \boldsymbol{\nabla}(\partial_5\tilde{\phi}) = \mathbf{0} \\
&\boldsymbol{\nabla}(\partial_4\tilde{\phi}) = \partial_4(\boldsymbol{\nabla}\tilde{\phi}) = \frac{1}{v_4}\boldsymbol{\nabla}(\partial_t\phi).
\end{aligned}
\tag{21}
$$

This leads to equation (18). Note that whereas the field $\tilde{\phi}(x)$ is a Galilei scalar, the field $\phi(\boldsymbol{x}, t)$ transforms as the fifth component of a Galilei vector. Indeed under a Galilei transformation we have

$$\begin{aligned} \tilde{\phi}'(x') &= \phi'(\boldsymbol{x}', t') + a_0 s' \\ &= \phi'(\boldsymbol{x}', t') + a_0 s - a_0 \boldsymbol{v} \cdot \boldsymbol{x} + a_0 \tfrac{1}{2} v^2 t \\ &= \phi(\boldsymbol{x}, t) + a_0 s = \tilde{\phi}(x) \end{aligned} \tag{22}$$

if the field ϕ transforms as

$$\phi'(\boldsymbol{x}', t') = \phi(\boldsymbol{x}, t) + a_0 \boldsymbol{v} \cdot \boldsymbol{x} - a_0 \tfrac{1}{2} v^2 t. \tag{23}$$

Then the Lagrangian takes the form

$$\mathcal{L} = \frac{\rho_0}{2v_0^2} \left(\frac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi - a_0 \frac{v_5}{v_4} \partial_t \phi - v_0^2 \right)^2 \tag{24}$$

and the equations of motion, equation (14), become

$$v_0^2 \nabla^2 \phi - a_0^2 \frac{v_5^2}{v_4^2} \partial_t^2 \phi = f(\phi) \tag{25}$$

where

$$f(\phi) = \nabla^2 \phi \left(\frac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi - \frac{a_0 v_5}{v_4} \partial_t \phi \right) + \frac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi) - 2 \frac{a_0 v_5}{v_4} \boldsymbol{\nabla}(\partial_t \phi) \cdot \boldsymbol{\nabla}\phi. \tag{26}$$

Observe that by neglecting the non-linear terms in equation (25), i.e. with $f(\phi) = 0$, one obtains the wave equation with velocity $\left|\frac{v_4 v_0}{a_0 v_5}\right|$. However, the non-linear terms ensure Galilean invariance of the equation.

The equation of motion can be expressed as a continuity equation:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \tag{27}$$

with the current density given as

$$\boldsymbol{J} = \rho \boldsymbol{v} = -\frac{v_4}{a_0 v_5} \rho \boldsymbol{\nabla}\phi \tag{28}$$

so one has

$$\boldsymbol{v} \equiv -\frac{v_4}{a_0 v_5} \boldsymbol{\nabla}\phi. \tag{29}$$

Thus equation (27) can be expressed in terms of the density ρ:

$$\partial_t \rho - \frac{v_4}{a_0 v_5} \boldsymbol{\nabla}\rho \cdot \boldsymbol{\nabla}\phi - \frac{v_4}{a_0 v_5} \rho \nabla^2 \phi = 0 \tag{30}$$

with

$$\rho = -\frac{\partial \mathcal{L}}{\partial \partial_t \phi} = \frac{\rho_0 a_0 v_5}{v_0^2 v_4} \left(\frac{1}{2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi - a_0 \frac{v_5}{v_4} \partial_t \phi - v_0^2 \right). \tag{31}$$

If we define

$$a_0 = -1 \tag{32}$$

which is suggested by requiring that the field ϕ of equation (23) transforms exactly as the component s in equation (1), as well as

$$v_4 = v_5 \tag{33}$$

then equation (25) becomes

$$v_0^2\nabla^2\phi - \partial_t^2\phi = \nabla^2\phi\left(\tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \partial_t\phi\right) + \tfrac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi\right) + 2\boldsymbol{\nabla}(\partial_t\phi)\cdot\boldsymbol{\nabla}\phi \tag{34}$$

which is the equation (5.40) obtained by Takahashi in [5]. Note that other values of a_0 are possible; for instance $a_0 = +1$ corresponds to the field transforming under a Galilean boost towards the negative direction. By neglecting the non-linear terms, i.e. the right-hand side, we obtain the wave equation with velocity $|v_0|$. With these choices the Lagrangian reduces to

$$\mathcal{L} = \frac{\rho_0}{2v_0^2}\left(\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \partial_t\phi - v_0^2\right)^2 \tag{35}$$

and the density becomes

$$\rho = -\frac{\rho_0}{v_0^2}\left(\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \partial_t\phi - v_0^2\right). \tag{36}$$

The velocity field reduces to the familiar expression $\boldsymbol{v} = \boldsymbol{\nabla}\phi$.

With the embedding defined in equation (20) and noting that

$$\partial^4\tilde{\phi} = -\partial_5\tilde{\phi} = -a_0v_5 \qquad \text{and} \qquad \partial^5\tilde{\phi} = -\partial_4\tilde{\phi} = -\frac{1}{v_4}\partial_t\phi \tag{37}$$

the various components of the energy–momentum tensor, equation (15), have the form

$$\begin{aligned}
T^{ij} &= -\partial_i\phi\partial_j\phi(\pi-\rho_0) + \delta_{ij}\frac{v_0^2}{2\rho_0}(\pi-\rho_0)^2\\
T^{i4} &= a_0v_5\partial_i\phi(\pi-\rho_0)\\
T^{i5} &= \frac{1}{v_4}\partial_t\phi\partial_i\phi(\pi-\rho_0)\\
T^{44} &= -a_0^2v_5^2(\pi-\rho_0)\\
T^{45} &= -\frac{v_0^2}{2\rho_0}(\pi-\rho_0)^2 - a_0\frac{v_5}{v_4}\partial_t\phi(\pi-\rho_0)\\
T^{55} &= -\frac{1}{v_4}(\partial_t\phi)^2(\pi-\rho_0)
\end{aligned} \tag{38}$$

where the canonical momentum of ϕ is given as

$$\pi - \rho_0 = \frac{\rho_0}{v_0^2}\left(\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi - a_0\frac{v_5}{v_4}\partial_t\phi - v_0^2\right). \tag{39}$$

Then to get the flow equations, we express the components of $T^{\mu\nu}$ in terms of energy density, $\mathcal{E}(x)$, momentum density, $p_i(x) = J_i(x)$, energy flow, $T_{ij}(x)$, momentum flow, $J_i^{(e)}(x)$, and the density of mass, $J_0(x)$, as

$$\begin{aligned}
&T^{ij}(x) = T_{ij}(x) \qquad T^{i5}(x) = \frac{1}{v_4}J_i^{(e)}\\
&T^{i4}(x) = -a_0v_5p_i(x) = -a_0v_5J_i(x)\\
&T^{45}(x) = \mathcal{E}(x) \qquad T^{44}(x) = -a_0\frac{v_5}{v_4}J_0(x).
\end{aligned} \tag{40}$$

Then it is easy to check that the flow equations are satisfied:

$$\begin{aligned}
&\partial_tJ_0(x) + \partial_iJ_i(x) = 0\\
&\partial_tp_i(x) + \partial_jT_{ji}(x) = 0\\
&\partial_t\mathcal{E}(x) + \partial_iJ_i^{(e)}(x) = 0.
\end{aligned} \tag{41}$$

These equations may be cast into a form that give the basic equations of hydrodynamics. Note that T^{55} appears as an auxiliary component since it does not play any role in the balance equations; it can be seen simply as a remnant of the fifth dimension.

To conclude this section, note that the two other models discussed in [6] for the description of a barotropic irrotational fluid can be written in a Galilei-covariant form by using an embedding as in equation (18). The first is given by

$$\mathcal{L} = \frac{1}{\gamma}\rho_0 v_0^2 \left(-\frac{1}{v_0^2}\Phi(x)\right)^{\frac{\gamma}{\gamma-1}} \tag{42}$$

where $\Phi(x) = \frac{1}{2}\nabla\phi \cdot \nabla\phi + \partial_t\phi$, and it describes a fluid with pressure p proportional to ρ^γ. The other Lagrangian is

$$\mathcal{L} = \rho_0 v_0^2 \exp\left(-\frac{1}{v_0^2}\Phi(x)\right) \tag{43}$$

and leads to the barotropic condition $p = \text{constant} \times \rho$.

5. Solutions of Takahashi wave equations

In this section we investigate the solutions of equation (34) in one dimension (i.e. $\phi = \phi(x, t)$) and discuss briefly the three-dimensional case. We first perform an analysis of the symmetry Lie group of the wave equations. By the 'symmetry group of a system of differential equations', we mean the largest local Lie group of point transformations that acts on the independent and dependent variables of the system and leaves invariant its solution set. In other words, it transforms solutions into other solutions.

5.1. Lie analysis of the wave equations

The purpose of this section is to set up the notation and provide a brief review of the Lie analysis of differential equations such as are described with more detail e.g. in [19]. Consider a system of m partial differential equations of order k with p independent variables $\{x\}$ and q dependent variables $\{u\}$:

$$\Delta^a(x, u^{(k)}) = 0 \qquad a = 1, \ldots, m \tag{44}$$

with

$$x = (x_1, \ldots, x_p) \in \mathcal{X} \sim \boldsymbol{R}^p \tag{45}$$

and

$$u = (u^1, \ldots, u^q) \in \mathcal{U} \sim \boldsymbol{R}^q \tag{46}$$

where $u^{(k)} = \{u, u_{(1)}, \ldots, u_{(k)}\}$ denotes all the derivatives up to order k for all the u^l ($l = 1, \ldots, q$). We use the notation u_x, u_{xx} etc to denote the partial derivatives $\partial_x u$, $\partial_{xx} u$ etc. By definition, the *symmetry group G* of the system, equation (44), acts on the base space

$$\mathcal{M} \subseteq \mathcal{X} \times \mathcal{U} \tag{47}$$

as

$$\begin{aligned} G : \ & \mathcal{M} \to \mathcal{M} \\ & (x, u) \to (\tilde{x}, \tilde{u}) = (\Lambda_g(x, u), \Omega_g(x, u)) \end{aligned} \tag{48}$$

so equation (44) is invariant. The mappings Λ_g and Ω_g are diffeomorphisms of the base space $\mathcal{M}$ and $g \in G$ represents the parameters of the transformation.

We consider functions

$$\begin{aligned} f &: \mathcal{X} \to \mathcal{U} \\ u^l &= f^l(x_1, \ldots, x_p) \qquad l = 1, \ldots, q \end{aligned} \tag{49}$$

and their kth *prolongations*

$$\mathrm{pr}^{(k)} f(x) \equiv \{f, f_x, f_{xx}, \ldots, f_{x^k}\} \tag{50}$$

where f_{x^k} means $f_{x_{i1},\ldots,x_{ik}}$, i.e. kth-order derivatives. They give rise to the space $\mathcal{U}^{(k)} = \mathcal{U} \times \mathcal{U}_{(1)} \times \cdots \times \mathcal{U}_{(k)}$, where $\mathcal{U}_{(k)}$ denotes the space of all kth-order derivatives. This provides us with the kth-order *jet space*

$$\mathcal{M}^{(k)} = \mathcal{X} \times \mathcal{U}^{(k)}. \tag{51}$$

The action, equation (48), of the group G on the base space $\mathcal{M}$ naturally induces an action on the jet space $\mathcal{M}^{(k)}$. This leads to the kth-order prolongation of the group action

$$\mathrm{pr}^{(k)} G : \{x, f(x), f_x, \ldots, f_{x^k}\} \to \{\tilde{x}, \tilde{f}(\tilde{x}), \tilde{f}_{\tilde{x}}, \ldots, \tilde{f}_{\tilde{x}^k}\}. \tag{52}$$

Rather than looking directly for the (in general, non-linear) transformations in equation (48) that leave the system of equation (44) invariant, Lie's approach consists in studying the linear infinitesimal transformations

$$\begin{aligned} \tilde{x} &= x + \varepsilon\eta(x, u) + \mathcal{O}(\varepsilon^2) \\ \tilde{u} &= u + \varepsilon\varphi(x, u) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{53}$$

Therefore, rather than looking for the whole symmetry group G, we find its Lie algebra L realized in terms of the vector fields

$$\boldsymbol{v} = \sum_{i=1}^{p} \eta^i(x, u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u) \frac{\partial}{\partial u^l}. \tag{54}$$

The coefficients η^i and φ_l do not depend on the derivatives u_x, u_{xx} etc since we have restricted ourselves to point transformations. They are determined by the condition that $\boldsymbol{v}$ belongs to L if its kth prolongation $\mathrm{pr}^{(k)}\boldsymbol{v}$ annihilates the system equation (44) on its solution surface:

$$\mathrm{pr}^{(k)}\boldsymbol{v} \cdot \Delta^a(x, u^{(k)})|_{\Delta^b=0} \qquad a, b = 1, \ldots, m \tag{55}$$

where

$$\mathrm{pr}^{(k)}\boldsymbol{v} = \boldsymbol{v} + \sum_{l=1}^{q} \sum_{J} \psi_l^J(x, u^{(k)}) \frac{\partial}{\partial u_J^l}. \tag{56}$$

The multi-index $J = (j_1, \ldots, j_p)$ characterizes the higher-order derivatives of u^l. For instance j_i is the number of times u^l is differentiated with respect to x_i. We may use the notation

$$u_J^l = \frac{\partial^{|J|} u^l}{\partial x_1^{j_1} \cdots \partial x_p^{j_p}} \tag{57}$$

where $|J| = j_1 + \cdots + j_p$. The coefficients ψ_l^J can be obtained through the recursive formula

$$\psi_l^{J+J_i} = D_i \psi_l^J - \sum_{j=1}^{p} \big(D_i \eta^j(x, u)\big) = u_{J+J_j}^l \qquad |J| \geqslant 1 \tag{58}$$

starting from the coefficients of the first prolongation

$$\psi_l^{J_i} = D_i \varphi_l(x, u) - \sum_{j=1}^{p} \big(D_i \eta^j(x, u)\big) u_{J_j}^l. \tag{59}$$

The J_i are p-tuples with 1 at the ith position and zeros elsewhere, and D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_J u^l_{J+J_i} \frac{\partial}{\partial u^l_J} \qquad 0 \leqslant |J| \leqslant k. \tag{60}$$

More details, as well as examples, can be found in Olver's textbook and the references therein [19]. The algorithm is straightforward and has been implemented on computer; hereafter we use such a program [20], so we present only the vector fields underlying the symmetry Lie algebra and then proceed with the symmetry reduction of the equations and the investigation of their solutions.

5.2. One-dimensional wave equation

The one-dimensional version of equation (34) is

$$v_0^2 \partial_{xx}\phi - \partial_{tt}\phi = \tfrac{3}{2}\partial_{xx}\phi(\partial_x\phi)^2 + \partial_{xx}\phi\partial_t\phi + 2\partial_{xt}\phi\partial_x\phi. \tag{61}$$

In terms of equations (44)–(46), here we have $m = 1$, $p = 2$, $q = 1$ and $k = 2$. The symmetry analysis described in the previous section provides the vector fields, equation (54):

$$\begin{aligned}
\boldsymbol{v}_1 &= \partial_t \\
\boldsymbol{v}_2 &= \partial_\phi \\
\boldsymbol{v}_3 &= \partial_x \\
\boldsymbol{v}_4 &= \left(t - \frac{\phi}{v_0^2}\right)\partial_\phi - \frac{x}{2v_0^2}\partial_x \\
\boldsymbol{v}_5 &= x\partial_\phi + t\partial_x \\
\boldsymbol{v}_6 &= x\partial_x + t\partial_t + \phi\partial_\phi.
\end{aligned} \tag{62}$$

Each one of them generates a one-parameter subgroup of equation (61) through the action of $\exp(\varepsilon\boldsymbol{v}_i)$ on the space of variables (x, t, ϕ), with ε being a real number which parametrizes the group. The non-zero commutation relations among these vector fields are

$$\begin{array}{lll}
[\boldsymbol{v}_1, \boldsymbol{v}_4] = \boldsymbol{v}_2 & [\boldsymbol{v}_1, \boldsymbol{v}_5] = \boldsymbol{v}_3 & [\boldsymbol{v}_1, \boldsymbol{v}_6] = \boldsymbol{v}_1 \\
[\boldsymbol{v}_2, \boldsymbol{v}_4] = -\dfrac{1}{v_0^2}\boldsymbol{v}_2 & [\boldsymbol{v}_2, \boldsymbol{v}_6] = \boldsymbol{v}_2 & [\boldsymbol{v}_3, \boldsymbol{v}_4] = -\dfrac{1}{2v_0^2}\boldsymbol{v}_3 \\
[\boldsymbol{v}_3, \boldsymbol{v}_5] = \boldsymbol{v}_2 & [\boldsymbol{v}_3, \boldsymbol{v}_6] = \boldsymbol{v}_3 & [\boldsymbol{v}_4, \boldsymbol{v}_5] = \dfrac{1}{2v_0^2}\boldsymbol{v}_5.
\end{array} \tag{63}$$

This is a solvable and non-nilpotent Lie algebra. If $\phi = f(x, t)$ is a solution of equation (61), then the subgroups generated by the vector fields in equation (62) imply that so are the functions

$$\begin{aligned}
\phi^{(1)} &= f(x, t - \varepsilon) \\
\phi^{(2)} &= f(x, t) + \varepsilon \\
\phi^{(3)} &= f(x - \varepsilon, t) \\
\phi^{(4)} &= v_0^2 t\left(1 - e^{-\varepsilon/v_0^2}\right) + e^{-\varepsilon/v_0^2} f(e^{\varepsilon/2v_0^2}x, t) \\
\phi^{(5)} &= \varepsilon x - \tfrac{1}{2}\varepsilon^2 t + f(x - \varepsilon t, t) \\
\phi^{(6)} &= e^{\varepsilon} f(e^{-\varepsilon}x, e^{-\varepsilon}t)
\end{aligned} \tag{64}$$

respectively. Moreover this implies that new solutions can be obtained from known solutions. For each such one-parameter subgroup of the full symmetry group there exists a class of group-invariant solutions which can be found from a reduced ordinary differential equation, which depends on the subgroup selected (see [19], chapter 3). Some important examples are analysed in the following subsections.

Note that in the limit where v_0^2 approaches infinity, the equations (34) and (61) become the Laplace equation, so propagation is lost and the fluid becomes static. From the point of view of the vector fields in equation (62), $\boldsymbol{v}_4$ reduces to $t\partial_\phi$ and the Lie algebra of equation (63) is replaced by the Lie algebra for which the commutators $[\boldsymbol{v}_2, \boldsymbol{v}_4]$, $[\boldsymbol{v}_3, \boldsymbol{v}_4]$ and $[\boldsymbol{v}_4, \boldsymbol{v}_5]$ vanish. Correspondingly the transformed function $\phi^{(4)}$ is reduced to

$$\phi^{(4)} = f(x,t) + \varepsilon t. \tag{65}$$

5.2.1. Travelling-wave solutions. These are group-invariant solutions arising from a symmetry subgroup which is a translation group on the space of independent variables. The associated vector field is

$$\boldsymbol{v} = \boldsymbol{v}_1 + c\boldsymbol{v}_3 = \partial_t + c\partial_x \tag{66}$$

where c is a constant. Group invariants of this subgroup include

$$y = x - ct \qquad v = \phi \tag{67}$$

since $\boldsymbol{v}y = 0 = \boldsymbol{v}v$. This allows a search for solutions of the form $\phi = f(x-ct)$, or $v = f(y)$, so one has to replace (using the notation ϕ_y to denote the derivative of ϕ with respect to y etc)

$$\partial_x\phi = \phi_y \qquad \partial_t\phi = -c\phi_y \tag{68}$$

in equation (61) to obtain

$$\left(v_0^2 - c^2 - \tfrac{3}{2}(\phi_y)^2 - 3c\phi_y\right)\phi_{yy} = 0. \tag{69}$$

This is satisfied whenever ϕ_{yy} or the expression in the parenthesis vanishes. The first possibility implies that ϕ is linear in $y = x - ct$ with the result that

$$\phi(x,t) = k_1(x - ct) + k_2 \tag{70}$$

is a (travelling-wave) solution of equation (61). If we consider the case where the expression in the parenthesis of equation (69) vanishes, we have the solution

$$\phi_y = -c \pm \sqrt{\frac{c^2 + 2v_0^2}{3}} = \text{constant}. \tag{71}$$

Integrating over y and substituting in terms of x and t, we obtain another solution:

$$\phi(x,t) = \left(-c \pm \sqrt{\frac{c^2 + 2v_0^2}{3}}\right)(x - ct) + k \tag{72}$$

where k is an arbitrary constant. This solution is a rather trivial solution of equation (61) since the second-order derivatives all vanish identically. Note also that this solution is identical to equation (70).

Note that another travelling-wave solution is obtained by considering the subgroup generated by

$$\boldsymbol{v} = \boldsymbol{v}_3 + c\boldsymbol{v}_2 = \partial_x + c\partial_\phi. \tag{73}$$

It admits the invariants

$$y = t \qquad v = \phi - cx \tag{74}$$

with c a non-zero constant. Then equation (61) is reduced to $v_{yy} = 0$, with the solution $v = k_1 y + k_2$, so

$$\phi(x,t) = k_1 t + cx + k_2. \tag{75}$$

5.2.2. Scale-invariant solutions. Here we consider the subgroup generated by $\boldsymbol{v}_6$ and which acts as

$$\exp(\varepsilon \boldsymbol{v}_6)(x, t, \phi) = (\mathrm{e}^{\varepsilon} x, \mathrm{e}^{\varepsilon} t, \mathrm{e}^{\varepsilon} \phi). \tag{76}$$

Let us choose the two invariants as

$$y = \frac{x}{t} \qquad v = \frac{\phi}{t} \tag{77}$$

so that

$$\begin{aligned} &\partial_t \phi = -y v_y + v \qquad \partial_x \phi = v_y \\ &\partial_{tt} \phi = \frac{y^2}{t} v_{yy} \qquad \partial_{xt} \phi = -\frac{y}{t} v_{yy} \\ &\partial_{xx} \phi = \frac{1}{t} v_{yy}. \end{aligned} \tag{78}$$

Using these expressions, equation (61) becomes an ordinary first-order differential equation:

$$(v_y)^2 - 2y v_y + \tfrac{2}{3}\left(v + y^2 - v_0^2\right) = 0. \tag{79}$$

If we introduce a new dependent variable

$$w \equiv v - v_0^2 - \tfrac{1}{2} y^2 \tag{80}$$

then equation (79) becomes

$$(w_y)^2 + \tfrac{2}{3} w = 0 \tag{81}$$

which admits the general solution

$$w = -\frac{1}{6} y^2 \pm \mathrm{i}\sqrt{\frac{2}{3}} k y + k^2. \tag{82}$$

Substituting this expression back into equations (80) and (77) we find

$$\phi(x, t) = \frac{1}{3} \frac{x^2}{t} + (v_0^2 + k^2) t \pm \mathrm{i}\sqrt{\frac{2}{3}} k x. \tag{83}$$

5.2.3. Solutions invariant under $\boldsymbol{v}_5$. The two invariants are

$$y = t \qquad v = x^2 - 2t\phi \tag{84}$$

since $\boldsymbol{v}_5 y = 0 = \boldsymbol{v}_5 v$. From this, the partial derivatives become

$$\begin{aligned} &\partial_x \phi = \frac{x}{y} \qquad \partial_t \phi = \frac{v - x^2}{2y^2} - \frac{v_y}{2y} \\ &\partial_{xx} \phi = \frac{1}{y} \qquad \partial_{xt} \phi = -\frac{x}{y^2} \\ &\partial_{tt} \phi = \frac{x^2 - v}{y^3} + \frac{v_y}{y^2} - \frac{v_{yy}}{2y} \end{aligned} \tag{85}$$

so equation (61) is reduced to the ordinary equation

$$v_{yy} - \frac{1}{y} v_y + \frac{1}{y^2} v + 2v_0^2 = 0. \tag{86}$$

This can be solved by defining a new dependent variable

$$u = v/y \tag{87}$$

so that equation (86) reduces to

$$yu_{yy} + u_y + 2v_0^2 = 0 \tag{88}$$

which can be written as a first-order equation:

$$yw_y + w = 0 \tag{89}$$

by defining $w \equiv u_y + 2v_0^2$. Its solution is

$$w = k_1/y \tag{90}$$

so by solving equation (88) for u and then equation (87) for v we obtain

$$v = k_1 y \ln y - 2v_0^2 y^2 + k_2 y \tag{91}$$

where k_1 and k_2 are constants of integration. By substituting back into the second part of equation (84) we find

$$\phi(x,t) = \frac{x^2}{2t} - \frac{k_1}{2}\ln t + v_0^2 t - \frac{k_2}{2}. \tag{92}$$

5.2.4. Solutions invariant under $\boldsymbol{v}_4$. The interest of the one-parameter subgroup generated by the vector field $\boldsymbol{v}_4$ is that it involves v_0^2. The two invariants are

$$y = t \qquad v = \frac{\phi - v_0^2 t}{x^2}. \tag{93}$$

This leads to the substitutions

$$\begin{array}{lll} \phi = x^2 v + v_0^2 t & \phi_x = 2xv & \phi_t = x^2 v_y + v_0^2 \\ \phi_{xx} = 2v & \phi_{xt} = 2x v_y & \phi_{tt} = x^2 v_{yy} \end{array} \tag{94}$$

which reduce equation (61) to

$$v_{yy} + 10 v v_y + 12 v^3 = 0. \tag{95}$$

This equation is of the form discussed in section 14.31 of [21]: it corresponds to the 'case (i)' with $A = -10$, $C = -12$ and all the other coefficients equal to zero. A solution is given by

$$v = \frac{c}{y+k} \qquad c = 1/2 \text{ or } 1/3 \tag{96}$$

where k is a constant of integration. We find, by replacing this solution into equation (93), that

$$\phi(x,t) = \frac{cx^2}{t+k} + v_0^2 t \qquad c = 1/2 \text{ or } 1/3. \tag{97}$$

To summarize, let us note that all the group-invariant solutions involve combinations of terms linear in x and t, as well as $\frac{x^2}{t}$ and $\ln t$. Whereas the appearance of the linear terms is expected because every term of equation (61) contains a second-order derivative, the multiple occurrences of $\frac{x^2}{t}$ seem related to the fact that the field ϕ transforms as the spatial component s in equation (1) which in turn is often defined within the embedding as $\frac{x^2}{2t}$.

5.3. Three-dimensional wave equation

It is not our aim to perform a rigorous study of the three-dimensional equation (34). Indeed the Lie analysis represents a formidable task even for a computerized algorithm. Among the vector fields generating the symmetry group we clearly have the translations

$$\partial_x,\ \partial_y,\ \partial_z,\ \partial_t,\ \partial_\phi \tag{98}$$

since we can replace each variable by adding to it a constant without modifying the equation (34). The symmetry Lie algebra also admits the dilation vector field

$$x\partial_x + y\partial_y + z\partial_z + t\partial_t + \phi\partial_\phi \tag{99}$$

because multiplying the five variables by a same factor just brings a global multiplicative constant in front of the equation. The equation is also invariant under rotations about the three axes:

$$y\partial_z - z\partial_y \qquad z\partial_x - x\partial_z \qquad x\partial_y - y\partial_x. \tag{100}$$

Here let us just show how to investigate the group-invariant solutions generated by the last vector field, i.e. the rotations about the z-axis. We can take the invariants as

$$r = \sqrt{x^2 + y^2},\ z,\ t \text{ and } \phi. \tag{101}$$

The derivatives become

$$\begin{aligned} &\phi_x = \phi_r \frac{x}{r} \qquad \phi_{xx} = \frac{\phi_r}{r} + \left(\phi_{rr} - \frac{\phi_r}{r}\right)\left(\frac{x}{r}\right)^2 \\ &\phi_{xy} = \left(\phi_{rr} - \frac{\phi_r}{r}\right)\frac{xy}{r^2} \qquad \phi_{xz} = \phi_{rz}\frac{x}{r} \qquad \phi_{xt} = \phi_{rt}\frac{x}{r} \end{aligned} \tag{102}$$

the remaining derivatives involving y being obtained from those with respect to x by replacing x with y. From these expressions, equation (34) is reduced to an equation depending only on r, t and z:

$$\begin{aligned} (v_0^2 - 1)&\left(\phi_{rr} + \frac{\phi_r}{r} + \phi_{zz}\right) - \phi_{tt} - \frac{1}{2}\left(\phi_{rr} + \frac{\phi_r}{r} + \phi_{zz}\right)(\phi_r^2 + \phi_z^2) \\ &-\phi_{rr}\phi_r^2 - \phi_z^2\phi_{zz} - 2\phi_r\phi_z\phi_{rz} - 2\phi_r\phi_{rt} - 2\phi_z\phi_{zt} = 0. \end{aligned} \tag{103}$$

6. Concluding remarks

In this paper we have illustrated the use of a covariant-like formulation of Galilean invariance in five dimensions by investigating a model for compressible irrotational barotropic fluid with pressure proportional to the square of the density of mass. Other similar models have also been mentioned. The field $\phi(\boldsymbol{x}, t)$ is the velocity potential for the fluid moving with velocity $\boldsymbol{\nabla}\phi$ with respect to the observer. The equations obtained here can therefore be interpreted as describing the phonon in the fluid moving with this velocity.

Clearly the Galilei-covariant formalism can serve as a guide for constructing other non-relativistic field equations. Here let us just illustrate this point with two equations for fluids: the diffusion equation, and the Chaplygin gas model. The first one is obtained by considering a Lagrangian

$$\mathcal{L}_{\text{diff}} = \tfrac{1}{2}\epsilon\partial_\mu\tilde{\rho}\partial^\mu\tilde{\rho} - \tfrac{1}{2}f(x)\tilde{\rho}^2 \tag{104}$$

in terms of the field $\tilde{\rho}(x)$, where $f(x)$ is a function of the five-vector x. The Euler–Lagrange equation gives

$$\epsilon\partial^\mu\partial_\mu\tilde{\rho} = -f(x)\tilde{\rho}. \tag{105}$$

If we use the embedding given by equation (20) for x^4 and x^5 with $v_4 = v_5 = v_0$, define

$$\tilde{\rho}(x) = \exp\left(\frac{v_0}{2D}s\right)\rho(\boldsymbol{x}, t) \tag{106}$$

and choose $f(x) = f(\boldsymbol{x}, t)$, then equation (105) becomes the well-known diffusion equation, with diffusion coefficient D:

$$\partial_t\rho - D\boldsymbol{\nabla}^2\rho = \frac{D}{\epsilon}f(\boldsymbol{x}, t)\rho. \tag{107}$$

The Chaplygin model (see [4], or section 108 of Laudau and Lifshitz's textbook [12]) can be obtained from the Lagrangian density for two fields, $\tilde{\phi}$ and $\tilde{\rho}$:

$$\mathcal{L}_{\mathrm{Chapl}} = \tfrac{1}{2} j^{\mu} \partial_{\mu} \tilde{\phi} + V(\tilde{\rho}) \tag{108}$$

where j^{μ} is the mass current density, $\tilde{\rho}\partial^{\mu}\tilde{\phi}$ and $V(\tilde{\rho})$ is a potential which depends only on $\tilde{\rho}$. Let us consider the special case

$$V(\tilde{\rho}) = \frac{\lambda}{\tilde{\rho}} \qquad \lambda > 0 \tag{109}$$

which is the Chaplygin gas potential, with enthalpy $V' = -\frac{\lambda}{\tilde{\rho}^2}$, negative pressure $p = -\frac{2\lambda}{\tilde{\rho}}$ and speed of sound $\frac{\sqrt{2\lambda}}{\tilde{\rho}}$. We derive, by using the Euler–Lagrange equation for $\tilde{\rho}$ and the embedding of equation (20), the Bernouilli equation:

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = \frac{\lambda}{\rho^2}. \tag{110}$$

The equations of motion for $\tilde{\phi}$ give us the equation of continuity

$$\partial_t \rho + \nabla \cdot (\rho \nabla \phi) = 0. \tag{111}$$

We have taken $\tilde{\rho}(x) = \rho(\boldsymbol{x}, t)$. The Chaplygin equation is used to describe certain deformable bodies.

Finally, another equation of considerable interest in physics is the non-linear Schrödinger equation. It can be obtained from the manifestly Galilei-covariant Lagrangian

$$\mathcal{L}_{\mathrm{NLS}} \propto \partial_{\mu} \tilde{\psi} \partial^{\mu} \tilde{\psi}^{*} - \tfrac{1}{2} k (\tilde{\psi} \tilde{\psi}^{*})^{2} \tag{112}$$

with the embedding given by equation (20) for x^4 and x^5 but

$$\tilde{\psi}(x) = \mathrm{e}^{\mathrm{i}ms} \psi(\boldsymbol{x}, t) \tag{113}$$

for the field. This equation is related to the time-dependent Landau–Ginzburg theory of superconductivity [23] and has been used to describe the self-modulation of a monochromatic wave [24] and Langmuir waves in plasmas [25], among other applications.

Acknowledgments

We are grateful to NSERC (Canada) and to CNPq (Brazil) for financial support. We thank Professor Y Takahashi for discussions at various stages of this project.

References

[1] Streater R F 1995 *Statistical Dynamics: a Stochastic Approach to Non-Equilibrium Thermodynamics* (London: Imperial College Press)

[2] Balescu R 1997 *Statistical Dynamics: Matter out of Equilibrium* (London: Imperial College Press)

[3] Künzle H P and Duval C 1994 Relativistic and nonrelativistic physical theories on five-dimensional space-time *Semantical Aspects of Spacetime Theories* ed U Majer and H J Schmidt (Mannheim: BI-Wissenschaftsverlag) pp 113–29

[4] Jackiw R 2000 (A particle field theorist's) Lectures on (supersymmetric, non-Abelian) fluid mechanics (and d-branes) *Preprint* physics/0010042

This preprint is based on lectures delivered at the *Workshop on Strings, Duality and Geometry (Montreal, March 2000)* and the *Montreal Workshop on Integrable Models in Condensed Matter and Non-Equilibrium Physics (Montreal, June 2000)*.

[5] Takahashi Y 1988 Towards the many-body theory with the Galilei invariance as a guide. I *Fortschr. Phys.* **36** 63–81

[6] Takahashi Y 1988 Towards the many-body theory with the Galilei invariance as a guide. II *Fortschr. Phys.* **36** 83–96

[7] Omote M, Kamefuchi S, Takahashi Y and Ohnuki Y 1989 Galilean covariance and the Schrödinger equation *Fortschr. Phys.* **37** 933–50

[8] Santana A E, Khanna F C and Takahashi Y 1998 Galilei covariance and (4, 1)-de Sitter space *Prog. Theor. Phys.* **99** 327–36

de Montigny M, Khanna F C, Santana A E and Santos E S 1999 Poincaré gauge theory and Galilean covariance *Ann. Phys., NY* **277** 144–58

de Montigny M, Khanna F C and Santana A E 2001 Galilean covariance and applications in physics *Recent Research Developments in Physics* (Trivandrum: Transworld Research Network) pp 45–92

[9] Fuschich W I and Nikitin A G 1994 *Symmetries of Equations in Quantum Mechanics* (New York: Allerton)

[10] Lévy-Leblond J M 1971 Galilei group and Galilean invariance *Group Theory and Applications* vol 2, ed E M Loebl (New York: Academic) pp 221–99

[11] Soper D E 1976 *Classical Field Theory* (New York: Wiley) section 7.3

[12] Landau L D and Lifshitz E M 1959 *Fluid Mechanics* (New York: Pergamon) p 507

Abrikosov A A, Gorkov L P and Dzyaloshinskii I 1965 *Quantum Field Theory Methods in Statistical Physics* (New York: Pergamon) p 10

[13] Takahashi Y 1987 An invitation to a Galilei invariant world *Wandering in the Fields: Festschrift for Professor Kazahiko Nishijima on the Occasion of his Sixtieth Birthday* ed K Kwarabayashi and A Ukawa (Singapore: World Scientific) pp 117–27

Takahashi Y and Ropchan C 1987 The role of a Lorentz-like transformation in non-relativistic field theory *Can. J. Phys.* **65** 484–8

Takahashi Y and Ropchan C 1986 The first order Schrödinger equation, the symmetrization of the stress tensor and the spin–orbit coupling *Prog. Theor. Phys.* **76** 1187–97

Takahashi Y and Ropchan C 1987 The energy–mass relation in non-relativistic field theory and $E = mc^2$ *Prog. Theor. Phys.* **77** 229–31

Takahashi Y 1986 On the Schrödinger field *Rationale of Beings: Festschrift in Honor of Gyo Takeda* ed K Ishikawa, K I Kawazoe, H Matsuzaki and K Takahashi (Singapore: World Scientific) pp 76–102

Kashiwa T, Seto K and Takahashi Y 1992 Some results in Galilei-invariant field theories *Nuovo Cimento* B **107** 745–54

[14] Marmo G, Morandi G, Simoni A and Sudarshan E C G 1988 Quasi-invariance and central extensions *Phys. Rev.* D **37** 2196–205

[15] Enz U 1963 Discrete mass, elementary length and a topological invariant as a consequence of a relativistic invariant variational principle *Phys. Rev.* **131** 1392–4

Rosen N and Rosenstock H B 1952 The force between particles in a nonlinear field theory *Phys. Rev.* **85** 257–9

Skyrme T H R 1958 A nonlinear theory of strong interactions *Proc. R. Soc.* A **247** 260–78

Skyrme T H R 1961 Particle states of a quantized field *Proc. R. Soc.* A **262** 237–45

[16] Bean C P and de Blois R W 1959 Ferromagnetic domain wall as a pseudorelativistic entity *Bull. Am. Phys. Soc.* **4** 53

Enz U 1964 Die dynamic der blochshen wand *Helv. Phys. Acta* **37** 245–51

[17] Lebwohl P and Stephen M J 1967 Properties of vortex lines in superconducting barriers *Phys. Rev.* **163** 376–9

Scott A C 1967 Steady propagation on long Josephson junctions *Bull. Am. Phys. Soc.* **12** 308–9

Scott A C 1969 A nonlinear Klein–Gordon equation *Am. J. Phys.* **37** 52–61

Scott A C 1970 Propagation of flux on a long Josephson tunnel junction *Nuovo Cimento* B **69** 241–61

[18] Born M and Infeld L 1934 Foundations of a new field theory *Proc. R. Soc.* A **144** 425–51

Born M and Infeld L 1934 On the quantization of the new field equations (part I) *Proc. R. Soc.* A **147** 522–46

Born M and Infeld L 1935 On the quantization of the new field equations (part II) *Proc. R. Soc.* A **150** 141–66

Chellone D S 1972 Newman–Penrose conserved quantities in Born–Infeld electrodynamics *J. Phys. A: Math. Gen.* **5** 1545–9

Feenberg F 1935 On the Born–Infeld field theory of the electron *Phys. Rev.* **47** 148–57

Olsen S L 1972 On the quantization of the Born–Infeld theory *Lett. Nuovo Cimento* **5** 745–7

Porter J R 1972 Peeling and conservation laws in the Born–Infeld theory of electromagnetism *Proc. Camb. Phil. Soc.* **72** 319–24

[19] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[20] Head A K 1993 LIE, a PC program for Lie analysis of differential equations *Comput. Phys. Commun.* **71** 241–8

Head A K 1996 LIE, a PC program for Lie analysis of differential equations *Comput. Phys. Commun.* **96** 311–3

[21] Ince E L 1944 *Ordinary Differential Equations* (New York: Dover)

[22] de Montigny M, Khanna F C, Santana A E, Santos E S and Vianna J D M 2000 Galilean covariance and the Duffin–Kemmer–Petiau equation *J. Phys. A: Math. Gen.* **33** L273–8

de Montigny M, Khanna F C, Santana A E and Santos E S 2001 Galilean covariance and the non-relativistic

Bhabha equations *J. Phys. A: Math. Gen.* **34** 8901–17

[23] de Gennes P G 1966 *Superconductivity of Metals and Alloys* (New York: Benjamin) ch 6

[24] Taniuti T and Washimi H 1968 Self trapping and instability of hydromagnetic waves along the magnetic field in a cold plasma *Phys. Rev. Lett.* **21** 209–12

Asano N, Taniuti T and Yajima N 1969 Perturbation method for nonlinear wave modulation. II *J. Math. Phys.* **10** 2020–4

[25] Fried B D and Ichikawa Y H 1973 On the nonlinear Schrödinger equation for Langmuir waves *J. Phys. Soc. Japan* **34** 1073–82

Ichikawa Y H, Imamura T and Taniuti T 1972 Nonlinear wave modulation in collisionless plasma *J. Phys. Soc. Japan* **33** 189–97

Shimizu K and Ichikawa Y H 1972 Automodulation of ion oscillation modes in plasma *J. Phys. Soc. Japan* **33** 789–92